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RESONANT LYAPUNOV FAMILIES OF PERIODIC MOTIONS OF REVERSIBLE SYSTEMS†

V. N. TKHAI

Moscow

e-mail: tkhai@ccas.ru

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Local periodic motions of a reversible system in the neighbourhood of the zero equilibrium position are investigated. In the non-degenerate case, to every pair of pure imaginary roots $\pm\lambda_j$ there corresponds a symmetric Lyapunov family L_j, provided there is no resonance $\lambda_j + p\lambda_k = 0$ $(p \in \mathbf{N})$. The scenario of the disappearance of the family L_k as $\varepsilon \to 0$ (where ε is the resonance detuning) is investigated. It is shown that resonant symmetric Lyapunov families LR_α arise and constructive conditions are obtained for the existence of LR_α for both $\varepsilon = 0$ and $\varepsilon \neq 0$. When $p = 1$ the existence of two cycles is observed; the cycles are mutually symmetric about the fixed set of the reversible system and each is distant $O(\sqrt{\varepsilon})$ from the origin. For a reversible system written in the form that is standard for oscillation theory, in "amplitude-angle" variables, a general theorem is established according to which symmetric periodic motions exist in the structurally unstable case; the theorem is basic for investigating the families LR_α.
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1. PRELIMINARY REMARKS

In [1] we developed Lyapunov's idea of using a generating system containing a small parameter to study structurally unstable cases in the theory of periodic motions. In particular, a general existence theorem [1, Theorem 5] for periodic motions of a system of standard form was proved; necessary and sufficient conditions were obtained, on the assumption that the system of amplitude equations has no multiple roots. The theorem has been used to study cycles in various almost-resonance cases [2], in generic systems and in Lyapunov systems. For reversible systems, the theorem needs a natural supplement, because in that case, as a rule, the system of amplitude equations admits of a family of solutions. Some important special cases of the proposition for reversible systems were considered in [1].

When investigating local periodic motions of an autonomous system (a Lyapunov system, a reversible system, or a generic system) in the neighbourhood of an equilibrium position, it proves useful to adopt a method in which rescaling the problem reduces to the problem of continuing the motion with respect to a small parameter. One thus proves the existence of one-parameter families of periodic motions near zero, in Lyapunov systems [3] and reversible systems [4, 5], and investigates cycles in Lyapunov systems and generic systems [2]. In resonance systems [2, 4, 5] one has the structurally unstable case of the theory of periodic motions of a system with a small parameter.

Under certain restrictions, a smooth autonomous reversible system

$$\begin{aligned} \dot{\mathbf{u}} &= \mathbf{A}\mathbf{v} + \mathbf{U}(\mathbf{u}, \mathbf{v}) \\ \dot{\mathbf{v}} &= \mathbf{B}\mathbf{u} + \mathbf{V}(\mathbf{u}, \mathbf{v}); \quad \mathbf{u} \in \mathbf{R}^l, \quad \mathbf{v} \in \mathbf{R}^n (l \geq n) \\ \mathbf{U}(\mathbf{u}, -\mathbf{v}) &= -\mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \mathbf{V}(\mathbf{u}, -\mathbf{v}) = \mathbf{V}(\mathbf{u}, \mathbf{v}); \quad \mathbf{U}(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad \mathbf{V}(\mathbf{0}, \mathbf{0}) = \mathbf{0} \end{aligned} \tag{1.1}$$

(**A** and **B** are constant matrices, and **U** and **V** are non-linear terms) admits of Lyapunov families of periodic motions in the neighbourhood of zero [5]. They are symmetric with respect to the fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v}: \mathbf{v} = \mathbf{0}\}$ of the reversible system, form an $(l - n + 1)$-family and exist if: (a) the characteristic equation of the linear approximation has a pair of pure imaginary roots, (b) none of the other roots equals $\pm ip\omega$ $(p \in \mathbf{N})$, c) rank$\mathbf{B} = n$.

Do Lyapunov systems exist if one of these conditions is violated? Condition (a) cannot be dropped, because a Lyapunov family is by definition close to oscillations of a linear system due to a pair of pure imaginary roots. The case in which condition (b) fails to hold has been investigated for vector fields [6]. It was assumed there that dim$\mathbf{U}$ = dim$\mathbf{V}$, and multiple roots were considered on the assumption that a Jordan cell exists. Finally, some cases with rank$\mathbf{B} = n - 1$ have recently been considered [5],

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observing, in particular a "non-holonomic constraint" effect, namely, the dimension of the vector $\mathbf{u}$ exerts an influence on the existence of Lyapunov families.

Below, considering reversible systems formulated in the standard "amplitude-angle" variables of oscillation theory, we shall prove an existence theorem for symmetric periodic motions in the structurally unstable case. We shall then study the disappearance of Lyapunov families in almost-resonance situations. Finally, Lyapunov families will be investigated in the case of two-frequency resonances and for passage through resonance.

In all cases, constructive, verifiable conditions will be obtained for the existence of the desired families, in terms of the coefficients of the normal form of the system.

The problem of the existence of Lyapunov periodic motions is investigated for $l \geqslant n$. The cases of exact resonance and almost-resonance have both been analysed, and equations defining Lyapunov families have been obtained. According to part of a theorem of [5], when $l > n$ one can transform to the neighbourhood of a selected point of the manifold of equilibrium and obtain a problem involving motions in the neighbourhood of the "new" zero equilibrium position in an almost-resonance situation. Hence the problem of the evolution of resonant Lyapunov families on passing from one point of the manifold to another, and when the resonance detuning parameter passes through zero, is solved in a uniform manner.

In systems of general form, close to resonance systems, a cycle is produced as a rule at a distance $O(\varepsilon^\sigma)$ from the equilibrium position, with $\sigma = 1$ for 1:2 resonance and $\sigma = 1/2$ for 1:1 and 1:3 resonances [2]. In Lyapunov systems and Hamiltonian systems, cycles are produced at each level of the energy integral and they form a cycle of families (see [2] and [7, Chap. 8, Sec. 3.2]). Resonant Lyapunov families (with $\varepsilon = 0$) also exist in such systems [8–12].

In a reversible system, periodic motions exist both when $\varepsilon = 0$ and when $\varepsilon \neq 0$, forming resonant Lyapunov families near the equilibrium. This general rule is violated at 1:1 resonance. Here, along with a symmetric Lyapunov family, two cycles are produced, symmetric to one another relative to the fixed set, each distant $O(\sqrt{\varepsilon})$ from the equilibrium.

2. PERIODIC MOTIONS OF A REVERSIBLE SYSTEM WRITTEN IN STANDARD FORM

To investigate a system with a small parameter, when the generating system admits of a family of periodic motions, it is convenient to use standard notation for the system in "amplitude-angle" variables. In the case of a reversible system, the "amplitudes" and "angles" split into two groups of "amplitudes" and two groups of "angles". Finally, we note that it is often convenient to use two small parameters [1], while in the general case the rate of change of each "amplitude" is individual.

After these remarks, we express a reversible system as

$$\begin{aligned}
\dot{u}_\alpha &= \varepsilon^{p_\alpha} U_\alpha(\varepsilon, \mathbf{u}, \mathbf{v}, t) + \mu_\alpha(\mu) U_{1\alpha}(\varepsilon, \mu, \mathbf{u}, \mathbf{v}, t), \quad \alpha = 1, \dots, l_1 \\
\dot{v}_\beta &= \varepsilon^{q_\beta} V_\beta(\varepsilon, \mathbf{u}, \mathbf{v}, t) + \mu_\beta(\mu) U_{1\beta}(\varepsilon, \mu, \mathbf{u}, \mathbf{v}, t), \quad \beta = 1, \dots, n_1 \\
\dot{u}_\nu &= U_{0\nu}(\mathbf{u}, \mathbf{v}, t) + \varepsilon U_\nu(\varepsilon, \mathbf{u}, \mathbf{v}, t) + \mu U_{1\nu}(\varepsilon, \mu, \mathbf{u}, \mathbf{v}, t), \quad \nu = l_1 + 1, \dots, l \\
\dot{v}_\lambda &= V_{0\lambda}(\mathbf{u}, \mathbf{v}, t) + \varepsilon V_\lambda(\varepsilon, \mathbf{u}, \mathbf{v}, t) + \mu V_{1\lambda}(\varepsilon, \mu, \mathbf{u}, \mathbf{v}, t), \quad \lambda = n_1 + 1, \dots, n \\
U_j(\varepsilon, \mathbf{u}, -\mathbf{v}, -t) &= -U_j(\varepsilon, \mathbf{u}, \mathbf{v}, t) \\
U_{1j}(\varepsilon, \mu, \mathbf{u}, -\mathbf{v}, -t) &= -U_{1j}(\varepsilon, \mu, \mathbf{u}, \mathbf{v}, t), \quad j = 1, \dots, l \\
V_k(\varepsilon, \mathbf{u}, -\mathbf{v}, -t) &= V_k(\varepsilon, \mathbf{u}, \mathbf{v}, t) \\
V_{1k}(\varepsilon, \mu, \mathbf{u}, -\mathbf{v}, -t) &= V_{1k}(\varepsilon, \mu, \mathbf{u}, \mathbf{v}, t), \quad k = 1, \dots, n \\
U_{0\nu}(\mathbf{u}, -\mathbf{v}, -t) &= -U_{0\nu}(\mathbf{u}, \mathbf{v}, t), \quad V_{0\lambda}(\mathbf{u}, -\mathbf{v}, -t) = V_{0\lambda}(\mathbf{u}, \mathbf{v}, t) \\
\nu &= l_1 + 1, \dots, l; \quad \lambda = n_1 + 1, \dots, n \quad (l \geq n,\ l_1 \geq n_1)
\end{aligned} \tag{2.1}$$

($p_\alpha, q_\beta \in \mathbf{N}$, $\mu_{\alpha,\beta}(0) = 0$; $\alpha = 1, \dots, l_1$, $\beta = 1, \dots, n_1$). The right-hand sides of the equations are assumed to be 2π-periodic in t; ε, μ are small parameters.

Suppose, when $\varepsilon = 0$ and $\mu = 0$, the system

$$\begin{aligned} &\dot{\mathbf{u}}_2 = \mathbf{U}_0(\mathbf{A}, \mathbf{u}_2, \mathbf{0}, \mathbf{v}_2, t), \quad \dot{\mathbf{v}}_2 = \mathbf{V}_0(\mathbf{A}, \mathbf{u}_2, \mathbf{0}, \mathbf{v}_2, t), \quad \mathbf{A} = \text{const} \\ &\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \quad \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2); \quad \mathbf{u}_1 \in \mathbf{R}^{l_1}, \quad \mathbf{v}_1 \in \mathbf{R}^{n_1} \end{aligned} \tag{2.2}$$

admits of 2π-periodic motions $\mathbf{u}_2 = \varphi(\mathbf{A}, t)$, $\mathbf{v}_2 = \psi(\mathbf{A}, t)$ which are symmetric with respect to the fixed set $\{\mathbf{u}_2, \mathbf{v}_2 : \mathbf{v}_2 = \mathbf{0}\}$. Then the necessary and sufficient conditions for 2π-periodicity of a solution of the reversible system (2.1) which is symmetric with respect to the fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ are

$$\begin{aligned} &\upsilon_\beta(\varepsilon, \mu, \mathbf{u}^0, \mathbf{0}, \pi) = 0, \quad \beta = 1, \ldots, n_1 \\ &\upsilon_\lambda(\varepsilon, \mu, \mathbf{u}^0, \mathbf{0}, \pi) = 0, \quad \lambda = n_1 + 1, \ldots, n \end{aligned} \tag{2.3}$$

($\mathbf{u}^0$ is the initial value of the variable $\mathbf{u}$). In that case the first group of Eqs (2.3) is satisfied identically in $\mathbf{u}^0$ at $\varepsilon = 0$, $\mu = 0$. Therefore, in view of the fact that the rate of change of the variable υ_β is proportional to ε^{q_β}, we can write system (2.3) in the form

$$\begin{aligned} &\xi_\beta(\mathbf{u}^0) + \xi_{1\beta}(\varepsilon, \mathbf{u}^0) + \mu_\beta \varepsilon^{-q_\beta} f_\beta(\varepsilon, \mu, \mathbf{u}^0) = 0, \quad \beta = 1, \ldots, n_1 \\ &\eta_\lambda(\mathbf{u}^0) + \eta_{1\lambda}(\varepsilon, \mathbf{u}^0) + \mu_\lambda g_\lambda(\varepsilon, \mu, \mathbf{u}^0) = 0, \quad \lambda = n_1 + 1, \ldots, n \end{aligned} \tag{2.4}$$

where the functions $\xi_{1\beta}(\varepsilon, \mathbf{u}^0)$, $\eta_{1\lambda}(\varepsilon, \mathbf{u}^0)$ vanish when $\varepsilon = 0$. Hence it follows that if one chooses $\mu_\beta = o(\varepsilon^{q_\beta})$, $\mu = o(\varepsilon)$ as $\varepsilon \to 0$, then system (2.4) is solvable for sufficiently small $\varepsilon \neq 0$, provided the system of equations

$$\xi_\beta(\mathbf{u}^0) = 0, \quad \eta_\lambda(\mathbf{u}^0) = 0, \quad \beta = 1, \ldots, n_1, \quad \lambda = n_1 + 1, \ldots, n \tag{2.5}$$

is solvable and its solutions are such that

$$\operatorname{rank}\|\partial\xi_\beta/\partial\mathbf{u}^0, \partial\eta_\lambda/\partial\mathbf{u}^0\| = n \tag{2.6}$$

For any $\mathbf{u}_1^0 = \mathbf{A}$, the second group of Eqs (2.5), admits of a solution $\mathbf{u}_2^0 = \varphi(\mathbf{A}, \mathbf{0})$. Therefore the problem of finding the roots of Eqs (2.5) leads to a solvable system

$$\xi_\beta(\mathbf{A}, \varphi(\mathbf{A}, \mathbf{0})) = 0, \quad \beta = 1, \ldots, n_1 \tag{2.7}$$

The functions ξ_β are determined by integrating a system of differential equations

$$\dot{\xi}_\beta = V_\beta(0, \mathbf{A}, \varphi(\mathbf{A}, t), \mathbf{0}, \psi(\mathbf{A}, t), t), \quad \beta = 1, \ldots, n_1$$

over the interval $[0, \pi]$. Consequently, the roots of the equations are calculated from the system of amplitude equations

$$I_\beta(\mathbf{A}) = \int_0^\pi V_\beta(0, \mathbf{A}, \varphi(\mathbf{A}, t), \mathbf{0}, \psi(\mathbf{A}, t), t)dt = 0, \quad \beta = 1, \ldots, n_1 \tag{2.8}$$

To each root $\mathbf{A}^*$ of these equations that satisfies the condition $\operatorname{rank}\|\partial\eta_\lambda/\partial\mathbf{u}_2^0\| = n - n_1$ for $\mathbf{u}_1^0 = \mathbf{A}^*$, $\mathbf{u}_2^0 = \varphi(\mathbf{A}^*, \mathbf{0})$, there corresponds, for sufficiently small $\varepsilon \neq 0$, a solution of system (2.4), provided that

$$\operatorname{rank}\|\partial\mathbf{I}(\mathbf{A}^*)/\partial\mathbf{A}^*\| = n_1 \tag{2.9}$$

This solution depends on $l - n$ arbitrary parameters, of which $l_1 - n_1$ are chosen from the set $\{A_1, \ldots, A_{l_1}\}$; the remaining $k = l - n - (l_1 - n_1)$ determine, for every $\mathbf{A}$, a k-family of symmetric 2π-periodic solutions of system (2.2).

Thus, the existence of 2π-periodic motions in system (2.1) has been proved. These motions are described by formulae

$$u_\alpha = A_\alpha^* + \varepsilon^{p_\alpha}\int_0^t U_\alpha(0, \mathbf{A}^*, \boldsymbol{\varphi}(\mathbf{A}^*, t), \mathbf{0}, \boldsymbol{\psi}(\mathbf{A}^*, t), t)dt + o(\varepsilon^{p_\alpha}), \quad \alpha = 1, \ldots, l_1$$

$$\upsilon_\beta = \varepsilon^{q_\beta}\int_0^t V_\beta(0, \mathbf{A}^*, \boldsymbol{\varphi}(\mathbf{A}^*, t), \mathbf{0}, \boldsymbol{\psi}(\mathbf{A}^*, t), t)dt + o(\varepsilon^{q_\beta}), \quad \beta = 1, \ldots, n_1 \tag{2.10}$$

$$\mathbf{u}_2 = \boldsymbol{\varphi}(\mathbf{A}^*, t) + \mathbf{O}(\varepsilon), \quad \boldsymbol{\upsilon}_2 = \boldsymbol{\psi}(\mathbf{A}^*, t) + \mathbf{O}(\varepsilon)$$

and they form an $(l-n)$-family.

Theorem 1. To every root $\mathbf{A}^*$ of the amplitude equation (2.8) there corresponds a unique $(l-n)$-family (2.10) of symmetric 2π-periodic motions of system (2.1), provided that: (a) condition (2.9) holds, (b) the variational equations for the solution $\mathbf{u}_2 = \boldsymbol{\varphi}(\mathbf{A}, t)$, $\boldsymbol{\upsilon}_2 = \boldsymbol{\psi}(\mathbf{A}, t)$ of system (2.2) have for $\mathbf{A} = \mathbf{A}^*$ at most $l - n - (l_1 - n_1)$ roots of the characteristic equation equal to unity, (c) $\mu_\alpha = o(\varepsilon^{p_\alpha})$, $\mu_\beta = o(\varepsilon^{q_\beta})$, $\mu = o(\varepsilon)$ as $\varepsilon \to 0$.

Corollary. **Cesari's method [13, 14]**. To every simple root A^* of the amplitude equation

$$\mathbf{I}(\mathbf{A}) = \int_0^\pi \mathbf{V}(0, \mathbf{A}, \mathbf{0}, t)dt = 0$$

of a reversible system, 2π-periodic in t,

$$\dot{\mathbf{u}} = \mu\mathbf{U}(\mu, \mathbf{u}, \mathbf{v}, t), \quad \dot{\mathbf{v}} = \mu\mathbf{V}(\mu, \mathbf{u}, \mathbf{v}, t); \quad \mathbf{u}, \mathbf{v} \in \mathbf{R}^n$$

$$\mathbf{U}(\mu, \mathbf{u}, -\mathbf{v}, -t) = -\mathbf{U}(\mu, \mathbf{u}, \mathbf{v}, t), \quad \mathbf{V}(\mu, \mathbf{u}, -\mathbf{v}, -t) = \mathbf{V}(\mu, \mathbf{u}, \mathbf{v}, t)$$

with a small parameter μ there corresponds a 2π-periodic solution, close to a constant and symmetric with respect to the fixed set $\{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$:

$$\mathbf{u} = \mathbf{A}^* + \mu\int_0^t \mathbf{U}(\mathbf{0}, \mathbf{A}^*, \mathbf{0}, t)dt + o(\mu), \quad \mathbf{v} \equiv \mu\int_0^t \mathbf{V}(\mathbf{0}, \mathbf{A}^*, \mathbf{0}, t)dt = o(\mu)$$

Remark. In the case when system (2.1) is 2π-periodic in only part of the variables, the theorem establishes the existence of 2π-periodic rotational solutions (see [1, 15]).

3. THE DISAPPEARANCE OF ONE OF THE TWO LYAPUNOV FAMILIES AS THE SYSTEM APPROACHES A RESONANCE SYSTEM

In the case rank$\mathbf{B} = n$, system (1.1) may be reduced by a non-singular linear transformation to the following form [5]

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{P}\mathbf{y} + \boldsymbol{\Xi}(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \boldsymbol{\xi} \in \mathbf{R}^{l-n} \\ \dot{\mathbf{x}} &= \mathbf{J}\mathbf{y} + \mathbf{X}(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}) \\ \dot{\mathbf{y}} &= \mathbf{x} + \mathbf{Y}(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^n \end{aligned} \tag{3.1}$$

($\mathbf{P}$ is a constant matrix, $\mathbf{J}$ is a real Jordan matrix with real eigenvalues $\lambda_1^2, \ldots, \lambda_n^2$ – the squares of the roots of the characteristic equation, and Ξ, $\mathbf{X}$, $\mathbf{Y}$ are non-linear terms). Let us assume that there are two pairs of pure imaginary roots and consider a situation close to the case in which there is a single two-frequency resonance

$$\lambda_1 + p\lambda_2 = i\kappa\varepsilon, \quad \kappa = \text{const}, \quad p \in \mathbf{N} \tag{3.2}$$

(ε is the resonance detuning). The remaining roots satisfy condition (b) in system (1.1). Then, by a theorem in [5], system (3.1) with $\varepsilon \neq 0$ always admits of two Lyapunov families, but if $\varepsilon = 0, p > 1$, only the family corresponding to the root λ_1 exists; the conditions for the existence of the second family fail to hold. If $p = 1$ and $\varepsilon = 0$, the conditions for the existence of both families indicated above for $\varepsilon \neq 0$ do not hold.

Suppose $\varepsilon \neq 0$. We separate out the variables corresponding to the roots $\pm\lambda_1$ and $\pm\lambda_2$ and, making the replacement of variables $(\boldsymbol{\xi}, \mathbf{x}, \mathbf{y}) \to (\boldsymbol{\xi}, \mu\mathbf{x}, \mu\mathbf{y})$, introduce a small parameter μ. We obtain

$$\begin{aligned}
&\dot{\boldsymbol{\xi}} = \mu\mathbf{P}\mathbf{y} + \mu\boldsymbol{\Xi}^*(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y})\\
&\dot{x}_1 = \lambda_1^2 y_1 + \mu X_1^*(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \dot{y}_1 = x_1 + \mu Y_1^*(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y})\\
&\dot{x}_2 = \lambda_2^2 y_2 + \mu X_2(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \dot{y}_2 = x_2 + \mu Y_2^*(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y})\\
&\dot{\mathbf{x}}_s = \mathbf{J}_s\mathbf{y}_s + \mu\mathbf{X}_s(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad \dot{\mathbf{y}}_s = \mathbf{x}_s + \mu\mathbf{Y}_s^*(\mu, \boldsymbol{\xi}, \mathbf{x}, \mathbf{y}), \quad s = 3, \dots, s^*; \quad s^* \leq n
\end{aligned} \tag{3.3}$$

($\mathbf{x}_s$ and $\mathbf{y}_s$ are vectors with the dimensionality of the Jordan cell $\mathbf{J}_s$ corresponding to the eigenvalue λ_s^2). If $\mu = 0$, system (3.3) admits of a family of periodic solutions symmetric with respect to the fixed set $\{\boldsymbol{\xi}, \mathbf{x}, \mathbf{y} : \mathbf{y} = \mathbf{0}\}$,

$$\begin{aligned}
&\boldsymbol{\xi} = \boldsymbol{\xi}^0 \ (\boldsymbol{\xi}^0 = \text{const})\\
&x_\alpha = \omega_\alpha a_\alpha \cos\omega_\alpha t, \quad y_\alpha = a_\alpha \sin\omega_\alpha t, \quad \lambda_\alpha = -(-1)^\alpha i\omega_\alpha(\omega_\alpha > 0), \quad a_\alpha = \text{const}; \quad \alpha = 1, 2\\
&x_\beta = \sum_{j=3}^n a_j^* \varphi_{\beta j}(t), \quad y_\beta = \sum_{j=3}^n a_j \psi_{\beta j}(t); \quad \varphi_{\beta j}(-t) = \varphi_{\beta j}(t), \quad \psi_{\beta j}(-t) = -\psi_{\beta j}(t)\\
&a_j^*, a_j = \text{const}, \quad \beta, j = 3, \dots, n
\end{aligned}$$

where, since, if $\varepsilon = 0$, there is only one resonance, we have

$$\det\|\psi_{\beta j}(\pi/\omega_{1,2})\| \neq 0 \tag{3.4}$$

The necessary and sufficient conditions for the existence of a symmetric $2T$-periodic solution of system (3.3) may be written in the form

$$\begin{aligned}
&a_\alpha \sin\omega_\alpha T + \mu y_\alpha^*(\mu, \boldsymbol{\xi}^*, a_1, \dots, a_n, T) = 0, \quad \alpha = 1, 2\\
&\sum_{j=3}^n a_j \psi_{\beta j}(T) + \mu y_\beta^*(\mu, \boldsymbol{\xi}^*, a_1, \dots, a_n, T) = 0, \quad \beta = 3, \dots, n
\end{aligned} \tag{3.5}$$

If $\mu = 0$, system (3.5) has two obvious solutions for the constants $a_1, \dots, a_n$ such that $a_\beta = 0$ ($\beta = 3, \dots, n$), and one of the a_αs is not zero ($T = \pi/\omega_a$). Therefore, taking conditions (3.5) into account, we obtain: if $\mu \neq 0$, system (3.5) has a solution such that

$$a_1 = O(1), \quad T = \pi/\omega_1 + O(\mu); \quad a_2, \dots, a_n = O(\mu)$$

(a_1 is an arbitrary number). Thus a Lyapunov family for the pair of roots $\pm\lambda_1$ will always exist.

If $\mu = 0$, system (3.5) also has a non-trivial solution $a_1 = 0, a_2 = 1, a_3 = \dots = a_n = 0, T = \pi/\omega_2$. Let us write the first two equations of system (3.4) in the form

$$\begin{aligned}
&(-1)^p a_1 \sin(p\omega_2\Delta T + \varepsilon T) + \mu y_1^*(\mu, \boldsymbol{\xi}^*, a_1, \dots, a_n, T) = 0\\
&-a_2 \sin(\omega_2\Delta T) + \mu y_2^*(\mu, \boldsymbol{\xi}^*, a_1, \dots, a_n, T) = 0, \quad \Delta T = T - \pi/\omega_2
\end{aligned} \tag{3.6}$$

Hence it follows that, if $\mu \neq 0, \varepsilon \neq 0$, system (3.5) has a solution such that

$$a_1 = O(\mu/\varepsilon), \quad a_2 = 1, \quad a_3 = O(\mu), \dots, a_n = O(\mu), \quad \Delta T = O(\mu) \tag{3.7}$$

The parameters ε and μ in Eqs (3.6) are independent of one another. Moreover, for fixed ε we have $a_1 = O(\mu)$. Thus, a solution (3.7) of system (3.5) exists when $\mu < \mu_0$, $\mu_0 = o(\varepsilon)$. Hence we infer that, together with ε, the amplitude of the periodic oscillations corresponding to the frequency ω_2 tends to zero: the Lyapunov family disappears.

Theorem 2. Suppose the characteristic equation of system (3.1) has two pairs of pure imaginary roots $\pm\lambda_1$, $\pm\lambda_2$, which satisfy relation (3.2), and the remaining roots are not multiples of λ_2. Then the maximum amplitude A_2 of oscillations on the Lyapunov family corresponding to the roots $\pm\lambda_2$ is $A_2 = O(\varepsilon)$, and as $\varepsilon \to 0$ this family contracts to the equilibrium $\boldsymbol{\xi} = \boldsymbol{\xi}^*$, $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ and disappears.

Remark. By a theorem in [5], system (3.2) has a manifold to equilibrium positions depending on $\boldsymbol{\xi}^*$, with a Lyapunov family near each. All the families corresponding to roots $\pm\lambda_2$ disappear as $\varepsilon \to 0$.

4. THIRD-ORDER 1:2 RESONANCE

In what follows, when considering specific resonances in system (3.1), attention will be confined to the sub-system corresponding to the resonant roots $\pm\lambda_1$, $\pm\lambda_2$.

We reduce system (3.1) to normal form up to terms of the required (second or third) order. To do this, as in [2], we apply to system (3.1), which is almost resonant, a normalizing transformation which is continuous in the parameter ε. We then introduce a small parameter μ using scaling.

In complex-conjugate variables z and $\bar{z}$ we obtain

$$\dot{z}_s = \lambda_s z_s + \mu i B_s \prod_{\alpha=1}^{2} \bar{z}_\alpha^{p_\alpha - \delta_{s\alpha}} + \mu^2 Z_s(\mu, \mathbf{z}, \bar{\mathbf{z}}), \quad s = 1, 2$$

$$\lambda_1 + 2\lambda_2 = i\kappa\varepsilon; \quad p_1 = 1, \quad p_2 = 2; \quad Z_s(\mu, \mathbf{0}, \mathbf{0}) = 0$$

(B_s are real constants). We consider the non-degenerate case, when $B_s \neq 0$ ($s = 1, 2$), and, using the formulae

$$z_s = \sqrt{|B_s| r_s} \exp(i\theta_s), \quad \bar{z}_s = \sqrt{|B_s| r_s} \exp(-i\theta_s); \quad s = 1, 2$$

transform to polar coordinates (r_s, θ_s). We obtain a reversible system

$$\begin{aligned} \dot{r}_s &= 2\mu B \operatorname{sign} B_s \sqrt{r_1} r_2 \sin\theta + o(\mu) \\ \dot{\theta}_s &= -i\lambda_2 + \frac{\mu B \operatorname{sign} B_s}{2 i r_s^{1/2}} \prod_{\alpha=1}^{2} r_\alpha^{p/2 - \delta_{s\alpha}} \cos\theta + o(\mu); \quad s = 1, 2 \\ B &= \sqrt{|B_1|} |B_2| \end{aligned} \tag{4.1}$$

which is invariant with respect to the substitution $(t, \mathbf{r}, \theta) \to (-t, \mathbf{r}, -\theta)$.

The last two equations for θ_s are replaced by a single equation for the angle $\theta = \theta_1 + 2\theta_2$:

$$\dot{\theta} = \kappa\varepsilon + \mu B\left(\operatorname{sign} B_1 r_1^{-1/2} r_2 + 2 \operatorname{sign} B_2 r_1^{1/2}\right)\cos\theta + o(\mu) \tag{4.2}$$

and the angle θ_2 is taken as the new independent variable. The resulting third-order system, which is periodic in θ and θ_2, has the obvious solution

$$r_s = r_s^0 \ (r_s^0 = \text{const}), \quad \theta = \kappa\varepsilon\theta_2 + \theta^0, \quad \theta^0 = \text{const}; \quad s = 1, 2 \tag{4.3}$$

If $\varepsilon = 0$, this solution will be a constant; if $\sin\theta^0 = 0$, it will be symmetric relative to the fixed set

$$\mathbf{M}_* = \{r_1, r_2, \theta, \theta_2 : \sin\theta = 0, \sin\theta_2 = 0\}$$

Suppose $\varepsilon = \mu^\sigma$, $\sigma \geqslant 1$. Then it follows from Theorem 1 that to the constant solutions of the amplitude equation

$$\kappa^* \sqrt{r_1} + B(\operatorname{sign} B_1 r_2 + 2 \operatorname{sign} B_2 r_1)\cos\theta = 0 \tag{4.4}$$

($\kappa^* = \kappa$ if $\sigma = 1$ and $\kappa^* = 0$ if $\sigma > 1$) there correspond for $\mu < \mu_1$ (where μ_1 is some finite number) a symmetric periodic solution

$$r_s = r_s^0 + O(\mu), \quad s = 1, 2, \quad \theta = \theta^0 + O(\mu) \tag{4.5}$$

($\theta^0 = 0$ or π). Taking into consideration the fact that in system (3.1)

$$\lambda_s = i[-(-1)^s\omega_s + \kappa_s\varepsilon], \quad \kappa_s = \text{const}; \quad s = 1, 2; \quad \kappa_1 + p\kappa_2 = \kappa \tag{4.6}$$

we obtain the form of the solution in system (3.1)

$$\begin{gathered} x_s = \mu[\omega_s - (-1)^s\kappa_s\mu^\sigma]a_s\cos\theta_s, \quad y_s = \mu a_s \sin\theta_s, \quad a_s^2 = |B_s| r_s^0 + O(\mu); \quad s = 1, 2 \\ \theta = \theta^0 + O(\mu), \quad \theta_1 = \theta - 2\theta_2, \quad \theta_2 = [-\omega_2 + \kappa_2\mu^\sigma + O(\mu)]t + \theta_2^0 \end{gathered} \tag{4.7}$$

(r_1^0 and r_2^0 are related by Eq (4.4)).

If $\kappa = 0$, we have exact resonance. In that case, system (4.1) is independent of the parameter ε. Formulae (4.7) define a resonant family of symmetric Lyapunov motions near equilibrium. As is obvious from the amplitude equation (4.4), such a family exists only in the neighbourhood of a stable equilibrium ($B_1B_2 <$ [16]). The resonant family is represented in the (r_1^0, r_2^0) plane by the straight line $r_2^0 = 2r_1^0$ (Fig. 1); the Lyapunov family corresponding to the roots $\pm\lambda_1$ lies on the abscissa axis.

If $\kappa \neq 0$, we have on every curve $\varepsilon = \mu^\sigma$ a periodic system which depends on μ. If $\mu < \mu_1$, system (3.1) also admits of the family of periodic motions (4.7). For each fixed μ system (3.1), containing $\varepsilon = \mu^\alpha$, has a family (4.7), which depends on one arbitrary parameter – the amplitude r_1^0.

Thus, for any $\varepsilon \geqslant 0$ we have a Lyapunov family near equilibrium. This is the basic difference between local periodic motions in a reversible system and in other systems – generic, Lyapunov and Hamiltonian.

The generation of a non-trivial periodic motion when $\varepsilon \neq 0$ has been observed in a generic system as a cycle; in Lyapunov and Hamiltonian systems one obtains a family of cycles at each energy level [2].

In Lyapunov and Hamiltonian systems, only for $\varepsilon = 0$ do resonant Lyapunov families exist near an equilibrium [8–12].

In the case when $\sigma > 1$ the amplitude equation (4.4) does not depend on ε and the family (4.7) exists only in the neighbourhood of a stable equilibrium. If $\sigma = 1$, the solution of Eq. (4.4) has a different form depending on whether there is a stable or unstable equilibrium.

Suppose a stable equilibrium is being considered ($B_1B_2 < 0$). In (r_1, r_2, θ) space the amplitude equation (4.4) has the form

$$r_2^0 = 2r_1^0 - \kappa^*\sqrt{r_1^0}/(B\cos\theta^0), \quad \sin\theta^0 = 0 \tag{4.8}$$

and admits of two families of solutions: $\theta^0 = 0$ and $\theta^0 = \pi$ (Fig. 2). If $\kappa^* = 0$ (exact resonance), the Lyapunov family is represented by the straight line $r_2^0 = 2r_1^0$ in the (r_1^0, r_2^0) plane. This family also exists

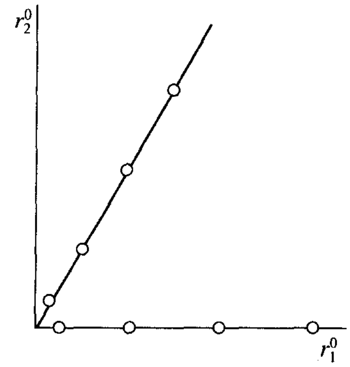

Fig. 1

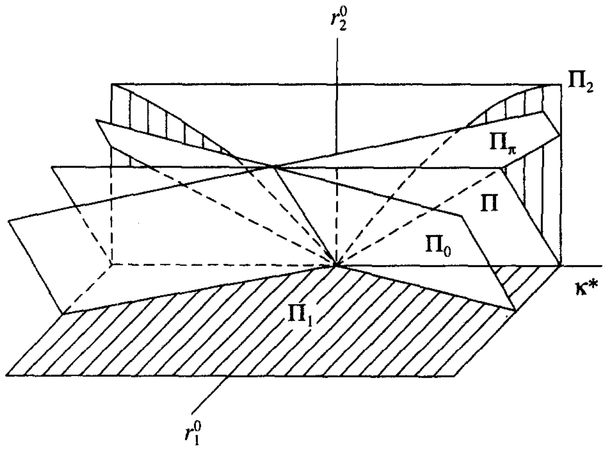

Fig. 2

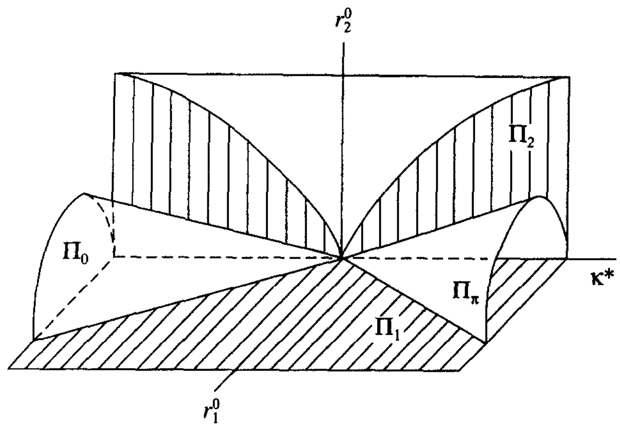

Fig. 3

if $\varepsilon > 0$ ($\varepsilon = \mu^\sigma$, $\sigma > 1$) and is represented by a plane Π in (r_1^0, r_2^0, κ^*) space. As before, it is near zero $r_1^0 = 0$, $r_2^0 = 0$. If $\kappa^* > 0$, the family $r_1^0 = 2r_2^0$ splits into three families, one of which is represented by the plane Π. The other two families are defined by Eqs (4.8). One of them, $\Pi_\pi(\theta^0 = \pi)$, is near the κ^* axis, the other, Π_0 ($\theta^0 = 0$), is near the points $\kappa^* = 2B\sqrt{r_1^0}$. In the (r_1^0, κ^*) plane there is a Lyapunov family corresponding to the roots $\pm\lambda_1$. This family bifurcates on the straight line $\kappa^* = 2B\sqrt{r_1^0}$ and becomes the family Π_0. Obviously, there is also a Lyapunov family in the (r_2^*, κ^*) plane if $\kappa^* > 0$, corresponding to $\pm\lambda_2$.

The non-resonant Lyapunov families, whose existence was established in [5], are shown in Fig. 2 by the hatching.

We will now consider the case of an unstable equilibrium ($B_1B_2 > 0$). In that case the amplitude equation is

$$r_2^0 = -2r_1^0 - \kappa^*\sqrt{r_1^0}/(B\cos\theta^0), \quad \sin\theta^0 = 0 \tag{4.9}$$

In the case of exact resonance there is clearly no Lyapunov family. A similar situation occurs when $\varepsilon = \mu^\sigma(\sigma > 1)$. If $\sigma = 1$, the family $\theta^0 = 0$ exists only for $\kappa^* < 0$, and the family $\theta^0 = \pi$ for $\kappa^* > 0$ (Fig. 3). The corresponding surfaces are situated below the (r_1^0, κ^*) plane, cutting it along the κ^* axis and the straight lines $\kappa^* = \pm 2B\sqrt{r_1^0}$. The families are near zero and simultaneously cut the family Π_1, corresponding to roots $\pm\lambda_1$. The family Π_1 bifurcates on the straight lines $\kappa^* = \pm B\sqrt{r_1^0}$. For every fixed $\kappa^*\varepsilon$ the family of Lyapunov motions is represented by points of a curve parallel to the (r_1^0, r_2^0) plane and contains, in particular, points near the equilibrium $r_1^0 = 0$, $r_2^0 = 0$. Oscillations occur at an amplitude $O(\varepsilon A)$, where A depends on the selected point of the curve (Fig. 3).

Theorem 3. In a reversible system with resonance $\lambda_1 + 2\lambda_2 = i\kappa\varepsilon$ (where κ = const and ε is a small non-negative number), depending on the signs of the number B_1 and B_2, the following Lyapunov families of periodic motions exist.

In the neighbourhood of a stable equilibrium ($B_1B_2 < 0$) with $\varepsilon = 0$ (exact resonance), we have a resonant family ($r_2^0 = 2r_1^0$) near the equilibrium. If $\kappa\varepsilon \neq 0$, this family splits into three surfaces, one of which corresponds to $\theta^0 = 0$ (Π_0), another to $\theta^0 = \pi(\Pi_\pi)$, and on the third, the plane Π, we have $r_2^0 = 2r_1^0$. Periodic oscillations on these surfaces are described by formulae (4.7) with $\varepsilon = \mu^\sigma$ ($\sigma > 1$) in the plane Π and $\varepsilon = \mu$ on the other surfaces. If $\kappa^* > 0$ (<0), the family Π_0 (Π_π) is separated from (near) the equilibrium. In the coordinate planes Lyapunov families Π_1 and Π_2 exist corresponding to pairs of roots $\pm\lambda_1$ and $\pm\lambda_2$, but the family Π_2 exists only if $\varepsilon \neq 0$.

In the neighbourhood of an unstable equilibrium ($B_1B_2 > 0$) there are no resonant Lyapunov families. If $\kappa^* < 0$, only the family Π_0 exists, and if $\kappa^* > 0$, only Π_π; the motions are described by formulae (4.7) with $\sigma = 1$. Families Π_1 and Π_2 exist as before.

5. THE STABILITY OF LOCAL PERIODIC MOTIONS AT 1:2 RESONANCE

Consider the following third-order system, which is periodic in θ_2 and θ

$$\frac{dr_s}{d\theta_2} = \frac{\mu}{\Phi_2}[2B\operatorname{sign}B_s\sqrt{r_1}r_2\sin\theta + \sqrt{r_s}R_s]$$

$$\frac{d\theta}{d\theta_2} = \frac{1}{\Phi_2}\left\{\kappa\mu^\sigma + \mu\left[B\left(\operatorname{sign}B_1\frac{r_2}{\sqrt{r_1}} + 2\operatorname{sign}B_2\sqrt{r_1}\right)\cos\theta + \frac{\Theta_1}{\sqrt{r_1}} + 2\frac{\Theta_2}{\sqrt{r_2}}\right]\right\} \tag{5.1}$$

$$\Phi_2 = -\omega_2 + \kappa_2\mu^\sigma + \mu\left(2B\operatorname{sign}B_2\sqrt{r_1} + \frac{\Theta_2}{\sqrt{r_2}}\right), \quad R_s, \Theta_s = o(\mu); \quad s = 1, 2$$

For a solution

$$\begin{gathered} r_s = r_s^0 + O(\mu), \quad s = 1, 2; \quad \theta = \theta^0 + O(\mu), \quad \sin\theta^0 = 0 \\ \kappa\mu^{\sigma-1}\sqrt{r_1^0} + 2B(\operatorname{sign}B_1 r_2^0 + \operatorname{sign}B_2 r_1^0) = 0 \end{gathered} \tag{5.2}$$

we formulate variational equations in the first approximation in μ

$$\begin{aligned} \frac{d(\Delta r_s)}{d\theta_2} &= -\frac{2\mu}{\omega_2}B\operatorname{sign}B_s\sqrt{r_1^0}r_2^0\cos\theta^0\Delta\theta \\ \frac{d(\Delta\theta)}{d\theta_2} &= -\frac{\mu B}{2r_1^0\sqrt{r_1^0}\omega_2}[(2\operatorname{sign}B_2 r_1^0 - \operatorname{sign}B_1 r_2^0)\Delta r_1 + 2\operatorname{sign}B_1 r_1^0\Delta r_2]\cos\theta^0 \end{aligned} \tag{5.3}$$

The characteristic equation of system (5.3) has a single zero root; two other roots are defined by the relation

$$\rho^2 = \mu^2B^2r_2^0(\omega_2 r_1^0)^{-1}(4r_1^0\operatorname{sign}B_1B_2 + r_2^0) \tag{5.4}$$

Hence it follows at once that in the neighbourhood of an unstable equilibrium ($B_1B_2 > 0$) all periodic motions of the families Π_0 and Π_π are unstable.

To evaluate ρ for a stable equilibrium ($B_1B_2 < 0$), we use (5.4), taking Eqs (4.4) into consideration. This gives

$$\rho^2 = -\mu^2B^2r_2^0(\omega_2\sqrt{r_1^0})^{-1}[2\sqrt{r_1^0} + \kappa^*/(B\cos\theta^0)]$$

Thus, at exact resonance the Lyapunov family is stable. It remains stable if $\varepsilon = \mu^{\sigma}$, $\sigma > 1$ (family Π). If $\sigma = 1$, then for $\kappa > 0$ the stable family is Π_0, but if $\kappa < 0$, the stable family is Π_{π}. Finally, depending on the combined influence in system (3.1) of the non-linearity and the magnitude of the detuning, we obtain: the family Π_0 (Π_{π}) is stable for $\kappa < 0$ ($\kappa > 0$) if $|\kappa| < 2B\sqrt{r_1^0}$; otherwise, it is unstable. These properties reflect the fact that the family of unstable periodic motions is near a stable zero.

Theorem 4. In the case of 1:2 resonance, the Lyapunov families in the neighbourhood of an unstable equilibrium ($B_1B_2 > 0$) are hyperbolic. If the equilibrium is stable, the resonant Lyapunov family is stable and transforms to stable families Π, $\Pi_0(\kappa > 0)$ and $\Pi_{\pi}(\kappa < 0)$. The families $\Pi_0(\kappa < 0)$, $\Pi_{\pi}(\kappa > 0)$ are stable if $|\kappa| > 2B\sqrt{r_1^0}$ and hyperbolic if $|\kappa| < 2B\sqrt{r_1^0}$.

Remarks. 1. A family is said to be stable if it consists of stable periodic motions.

2. The stability of solution (5.2) of Eq. (5.1) was investigated in the first approximation in the variables r_1, r_2, θ and the parameter μ. The conclusions of Theorem 3 as to hyperbolicity guarantee that the periodic motions will be unstable in Lyapunov's sense.

6. THE DEGENERATE CASE

The case usually encountered in mechanical problems is the degenerate one, $B = 0$. In that case the second-order terms in the normal form vanish, and system (3.1) in variables $\mathbf{z}$ and $\bar{\mathbf{z}}$ becomes

$$\dot{z}_s = \lambda_s z_s + i\mu^2[A_{s1}|z_1|^2 + A_{s2}|z_2|^2]z_s + \mu^3 Z_s(\mu, \mathbf{z}, \bar{\mathbf{z}}), \quad s = 1, 2 \tag{6.1}$$

(A_{sj} are real constants).

The normal form of a reversible system also has the form (6.1) in the case when there are no 1:1, 1:2 or 1:3 resonances. Consequently, we have a general problem concerning the family of Lyapunov periodic motions in degenerate cases for these resonances and for $1{:}p$, $p > 3$, resonances.

The amplitude equation in the cases of interest here may be written as follows:

$$\begin{gathered}\kappa^* + (A_1^* r_1 + A_2^* r_2) = 0, \quad \sin\theta^0 = 0, \quad A_s^* = A_{s1} + pA_{s2}, \quad s = 1, 2\\ \lambda_1 + p\lambda_2 = i\kappa\varepsilon \quad (\kappa = \text{const}), \quad p = 1, 2, \dots\end{gathered} \tag{6.2}$$

($\kappa^* = \kappa$ for $\varepsilon = \mu^2$ and $\kappa^* = 0$ for $\varepsilon = \mu^{\sigma+1}$, $\sigma > 1$).

We will consider two cases. If $A_1^* A_2^* < 0$, the surfaces of the Lyapunov families are shown in Fig. 4a. A resonant family ($\kappa^* = 0$) exists and is near zero. It is the intersection of families $\Pi(\varepsilon = \mu^{\sigma+1}$, $\sigma > 1)$ and $\Pi^*(\varepsilon = \mu^2)$, which exist both for $\theta^0 = 0$ and for $\theta^0 = \pi$.

In the case when $A_1^* A_2^* > 0$ (Fig. 4b, $A_2^* > 0$), there is no resonant family, and if $\kappa \neq 0$, there is only the family Π^*.

All periodic motions are described by formulae (4.7) with μ replaced by μ^2, and they are stable. The families Π_1 and Π_2 are shown, as before, by hatching. Theorem 1 implies the following theorem.

Theorem 5. In the degenerate cases of 1:1, 1:2 and 1:3 resonance, and at resonance $1{:}p$ ($p > 3$), the families of Lyapunov periodic motions are defined by the amplitude equation (6.2). If $A_1^* A_2^* < 0$, a resonant family exist, through which pass planes – the families Π and Π^*. If $A_1^* A_2^* > 0$, there is no resonant family, and if $\kappa^*/A_2^* < 0$, only the family Π^* exists.

7. FOURTH-ORDER 1:3 RESONANCE

In the case considered, the normal form of the system up to and including third-order terms, taking scaling into account, is

$$\dot{z}_s = \lambda_s z_s + i\mu^2\left[(A_{s1}|z_1|^2 + A_{s2}|z_2|^2)z_s + iB_s \prod_{\alpha=1}^{2} \bar{z}_{\alpha}^{p_\alpha - \delta_{s\alpha}}\right] + \mu^3 Z_s(\mu, \mathbf{z}, \bar{\mathbf{z}}), \quad s = 1, 2 \tag{7.1}$$

(A_{sj} and B_s are real numbers). As in Section 4, we consider the non-degenerate case $B_1B_2 \neq 0$ and introduce polar coordinates. Next, taking θ to be the angle $\theta = \theta_1 + 3\theta_2$ and taking the expression (4.5) with $p = 3$ into consideration, we can write down a third-order system, periodic in θ and θ_2

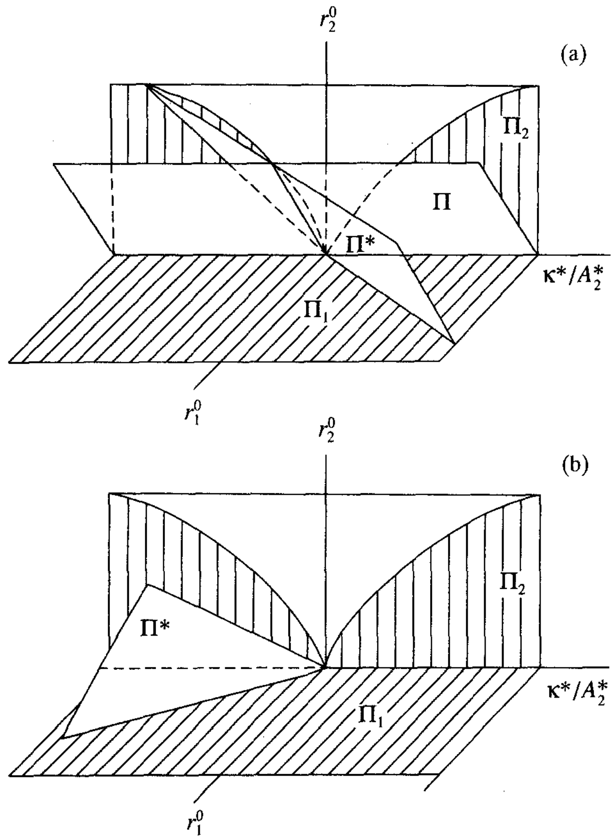

Fig. 4

$$\frac{dr_1}{d\theta_2} = \frac{1}{\Phi_2}\left\{2\mu^2 \operatorname{sign} B_s \sqrt{r_1 r_2^3} \sin\theta + O(\mu^3)\right\}$$

$$\frac{d\theta}{d\theta_2} = \frac{1}{\Phi_2}\left\{\kappa\varepsilon + \mu^2 B\left[A_1^* r_1 + A_2^* r_2 + \left(\operatorname{sign} B_1 r_1^{-1/2} r_2^{3/2} + 3\operatorname{sign} B_2 r_1^{1/2} r_2^{1/2}\right)\cos\theta + O(\mu^3)\right.\right. \qquad (7.2)$$

$$\Phi_2 = -\omega_2 + \kappa_2\mu^\sigma + \mu^2(A_{21}B_1 r_1 + A_{22}B_2 r_2 + 3B \operatorname{sign} B_2 r_1^{1/2} r_2^{1/2} \cos\theta) + O(\mu^3)$$

$$A_s = A_{s1} + 3A_{s2}, \quad A_s^* = A_s B_s / B, \quad B = \sqrt{|B_1||B_2|^3}$$

The amplitude equation for system (7.2) is

$$\kappa^* + B\left[A_1^* r_1^0 + A_2^* r_2^0 + (\operatorname{sign} B_1 r_2^0 + 3\operatorname{sign} B_2 r_1^0)\sqrt{\frac{r_2^0}{r_1^0}}\cos\theta^0\right] = 0, \quad \sin\theta^0 = 0$$

(the constant κ^* vanishes when $\varepsilon = \mu^{\sigma+1}(\sigma > 1)$ and $\kappa^* = \kappa$ when $\varepsilon = \mu^2$). By Theorem 1, Eq. (7.2) defines all the Lyapunov families at 1:3 resonance.

Theorem 6. In the case of 1:3 resonance, all Lyapunov families of symmetric periodic motions are defined by the amplitude equation (7.2). In the original system (3.1), these periodic motions are described by the formulae

$$\begin{aligned} &x_s = \mu[\omega_s - (-1)^s \kappa_s \mu^{\sigma+1}] a_s \cos\theta_s, \quad y_s = \mu a_s \sin\theta_s, \quad a_s^2 = |B_s| r_s^0 + O(\mu^2), \quad s = 1, 2 \\ &\theta = \theta^0 + O(\mu^2), \quad \theta_1 = \theta - 3\theta_2, \quad \theta_2 = [-\omega_2 + \kappa_2\mu^{\sigma+1} + O(\mu^2)]t + \theta_2^0 \end{aligned} \qquad (7.3)$$

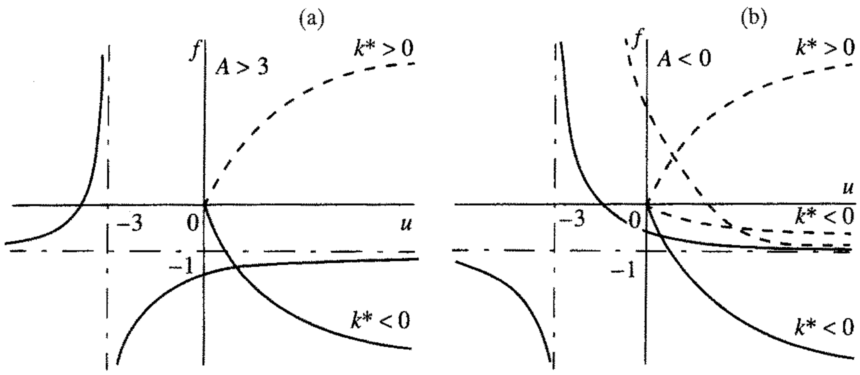

Fig. 5

(θ_2^0 = const, $\sigma \geqslant 1$; the constants κ_s are as defined in (4.6)). Depending on the coefficients A_1^*, A_2^*, B_1 and B_2, the families exist both at exact resonance ($\varepsilon = 0$) and if $\varepsilon \neq 0$. The conditions for these families to exist are the same for $\varepsilon = 0$ and $\varepsilon = \mu^{\sigma+1}(\sigma > 1)$, but for $\varepsilon = \mu^2$ we have $\kappa^* = \kappa$ in (7.2).

Let us analyse the conditions for Lyapunov families to exist when $\varepsilon = 0$ and $\varepsilon = \mu^{\sigma+1}(\sigma > 1)$. In that case $\kappa^* = 0$.

1. $B_1 B_2 > 0$. The amplitude equation (7.2) becomes

$$A_1^* + A_2^* u + (3+u)\sqrt{u} = 0, \quad \cos\theta^0 = 0, \quad \sin\theta^0 = 0, \quad u = r_2^0/r_1^0 \tag{7.4}$$

In the case when $A_2^* = 0$ a family always exists – but only one, defined by the equation

$$(3+u)\sqrt{u} = |A_1^*|$$

In the other special case, $|A_1^* + A_2^*| = 4$, the root of Eq. (7.4) is $u = 1$. This case corresponds to the boundary of the stable domain of the equilibrium [16] and yields a Lyapunov family $r_1^0 = r_2^0$ near equilibrium.

In the case when $A_2^* \neq 0$, we rewrite Eq. (7.4) as

$$\begin{gathered} f(u)\cos\theta^0 = k\sqrt{u} \\ f(u) = -(A+u)/(3+u), \quad k = 1/A_2^*, \quad A = A_1^*/A_2^* \end{gathered} \tag{7.5}$$

If $A = 3$, we have $u = |A_2^*|^2$, and there is a Lyapunov family. If $A \neq 3$, the relative positions of the graphs of the functions $f(u)$ and $k^*\sqrt{u}$ ($k^* = k\cos\theta^0$) are as shown in Fig. 5. Clearly, in the case when $A > 0$ (Fig. 5a) the graphs intersect at a single point (when $k^* < 0$), proving the existence of just one family, $\theta^0 = 0$ or $\theta^0 = \pi$.

If $0 < A < 3$, the zero of the function $f(u)$ lies to the left of the equilibrium (Fig. 5b) and the graphs intersect at only one point; a family with $k^* < 0$ exists. The case $A = 0$ is trivial. Here $A_1^* = 0, A_2^* \neq 0$. There is only one family. But if $A < 0$ (the curve $f(u)$ is shown in Fig. 5(b) by the dashed line), there is always an intersection (at $k^* > 0$). The other family appears when $k^* < 0$ as $|k^*|$ decreases.

If $B_1 B_2 > 0$, the equilibrium will be unstable if $|A_1^* + A_2^*| < 4$ [16]. It follows from the previous analysis that in that case there is at least one Lyapunov family.

2. $B_1 B_2 < 0$ (stable equilibrium). The amplitude equation is again as in (7.5), except that here $f(u) = -(A+u)/(3-u)$. It is obvious that, if $A = -3$, there is a single family $r_1^0 = r_2^0$. If $A \neq -3$, the relative positions of the graphs of the functions $f(u)$ and $k^*\sqrt{u}$ are as shown in Fig. 6. When $A > -3$ (Fig. 6a) there may be one ($k^* < 0$) or two ($k^* > 0$) families; one of the latter is not near the equilibrium if $\varepsilon \neq 0$. When $A < -3$ there is always one family (at $k^* < 0$), and depending on the values of the quantities A and $k^* > 0$, there may be one or two families (Fig. 6b).

The stability of the local periodic motions at 1:3 resonance is investigated on the basis of variational equations. For the solutions (r_1^0, r_2^0, θ^0) defined by the amplitude equation (7.2) we have

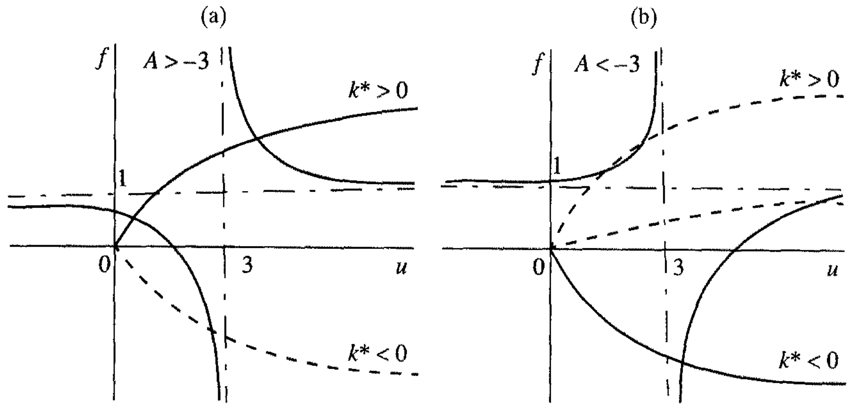

Fig. 6

$$\frac{d(\Delta r_s)}{d\theta_2} = \frac{2\mu^2}{\omega} B \operatorname{sign} B_s \sqrt{r_1^0 r_2^0}^{\,3} \cos\theta^0 \Delta\theta, \quad s = 1, 2$$

$$\frac{d(\Delta\theta)}{d\theta_2} = \frac{\mu^2}{\omega} B \left\{ A_1^* \Delta r_1 + A_2^* \Delta r_2 + \frac{1}{2}\left[\sqrt{\frac{r_2^0}{r_1^{0\,3}}} \operatorname{sign} B_1 (-r_2^0 \Delta r_1 + 3 r_1^0 \Delta r_2) + \right.\right. \tag{7.6}$$

$$\left.\left. + \frac{3}{\sqrt{r_1^0 r_2^0}} \operatorname{sign} B_2 (r_2^0 \Delta r_1 + r_1^0 \Delta r_2) \right] \cos\theta^0 \right\}; \quad \omega = \omega_2$$

The characteristic equation of system (7.6) has one zero root. The other two roots are $\pm\mu^2 B\omega^{-1}\rho_*$, where

$$\rho_*^2 = \sqrt{r_1^0 r_2^0}^{\,3} \left[2(A_1^* \operatorname{sign} B_1 + A_2^* \operatorname{sign} B_2)\cos\theta^0 + \right.$$

$$\left. + \frac{3}{\sqrt{u}} + 3\operatorname{sign}(B_1 B_2)\left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) - \sqrt{u}^{\,3} \right], \quad u = \frac{r_2^0}{r_1^0}$$

Consequently, if $\rho_* < 0$, we have a stable periodic motion, but if $\rho_* > 0$, the periodic motion is hyperbolic.

8. 1:1 RESONANCE

In the case of multiple roots with simple elementary divisors, as well as cases close to that of simple roots, the normal form is

$$\dot{r}_s = 2\mu^2[(B_{s1} r_1 + B_{s2} r_2)\sqrt{r_1 r_2} \sin\theta + b_s r_1 r_2 \sin 2\theta] + O(\mu^3)$$

$$\dot{\theta}_s = (-1)^{s-1}\omega + \kappa_s \varepsilon + \mu^2 \left[A_{s1} r_1 + A_{s2} r_2 + (B_{s1} r_1 + B_{s2} r_2)\sqrt{\frac{r_{3-s}}{r_s}} \cos\theta + b_s r_{3-s} \cos 2\theta \right] + O(\mu^3) \tag{8.1}$$

$$\kappa_1 + \kappa_2 = \kappa\varepsilon \quad (\kappa_{1,2} = \text{const}), \quad s = 1, 2$$

(A_{sj}, B_{sj}, b_s are constants, ε is the resonance detuning, and μ is a scaling factor). In this system all families of periodic motions, symmetric relative to the fixed set $\{r_1, r_2, \theta_1, \theta_2 : \sin\theta_1 = 0, \sin\theta_2 = 0\}$, are defined by the amplitude equation

$$\kappa^* + A_1^* r_1^0 + A_2^* r_2^0 + (B r_1^0 r_2^0 + B_{21} r_1^{0^2} + B_{12} r_2^{0^2})\frac{\cos\theta^0}{\sqrt{r_1^0 r_2^0}} = 0, \quad \sin\theta^0 = 0 \tag{8.2}$$

$$A_1^* = A_1 + b_2, \quad A_2^* = A_2 + b_1, \quad A_s = A_{1s} + A_{2s}, \quad s = 1, 2; \quad B = B_{11} + B_{22}$$

($\kappa^* = 0$ if $\varepsilon = 0$ or $\varepsilon = \mu^{\sigma+1}$, $\sigma > 1$; $\kappa^* = \kappa$ if $\varepsilon = \mu^2$). If $\kappa^* = 0$, this equation reduces to

$$(A_1^* + A_2^* u)\sqrt{u} + (B_{21} + Bu + B_{12}u^2)\cos\theta^0 = 0, \quad \sin\theta^0 = 0, \quad u = r_2^0/r_1^0 \tag{8.3}$$

Thus, the problem of the existence of symmetric Lyapunov families will be solved by finding the roots of Eq. (8.2) or Eq. (8.3).

The distinctive feature of 1:1 resonance is that a non-symmetric isolated periodic motion (cycle) may exist. In fact, if $\mu = 0$, the equations for r_1^0, r_2^0 have constant solutions in which $\sin\theta^0 \neq 0$, where θ^0 is determined from the condition $\dot{\theta} = 0$.

The system of amplitude equations [2] for determining non-symmetric periodic motions may be written in the form

$$B_{s1} r_1^0 + B_{s2} r_2^0 + b_s \sqrt{r_1^0 r_2^0}\cos\theta^0 = 0, \quad s = 1, 2, \quad \sin\theta^0 \neq 0$$
$$\kappa^* + A_1 r_1^0 + A_2 r_2^0 + (B r_1^0 r_2^0 + B_{21} r_1^{0^2} + B_{12} r_2^{0^2})\frac{\cos\theta^0}{\sqrt{r_1^0 r_2^0}} + (b_1 r_2^0 + b_2 r_1^0)\cos 2\theta^0 = 0 \tag{8.4}$$

Now, using the variable $x^2 = r_1^0/r_2^0$, we rewrite the first two equations of system (8.4) in the form

$$B_{s1} x^2 + b_s x \cos\theta^0 + B_{s2} = 0, \quad s = 1, 2$$

Next, multiplying these equations by b_2 and b_1, respectively, and subtracting one of the resulting equations from the other, we obtain

$$x^2 = -\frac{B_{12} b_2 - B_{22} b_1}{B_{11} b_2 - B_{21} b_1} > 0 \tag{8.5}$$

Then

$$\cos\theta^0 = -\frac{B_{12} B_{21} - B_{11} B_{22}}{B_{21} b_1 - B_{11} b_2} x^{-1} \tag{8.6}$$

Finally, it follows from the third equation of (8.4), taking relations (8.5) and (8.6) into consideration, that

$$r_2^0 = -\kappa^*/G, \quad G = A_1 x^2 + A_2 + (Bx + B_{21} x^2 + B_{12} x^{-1})\cos\theta^0 + (b_1 + b_2 x^2)\cos 2\theta^0 \tag{8.7}$$

It obvious from formula (8.7) that a cycle exists only if $\kappa^* = \kappa \neq 0$, that is, in an almost-resonance situation. Hence it also follows that $\varepsilon = \mu^2$, that is, a cycle will appear at a distance $\sqrt{\varepsilon}$ from zero. Finally, formula (8.15) implies the generation of two cycles on which $\theta^0 = \pm\theta^*$.

Theorem 7. In the case of 1:1 resonance of a reversible system, in the neighbourhood of an equilibrium, local symmetric periodic motions exist which form Lyapunov families. These families exist both at exact resonance ($\varepsilon = 0$) and at a distance $\varepsilon \neq 0$; all the families are defined by the amplitude equation (8.2). In addition, two cycles, symmetric with respect to one another relative to the fixed set, form in the neighbourhood $O\sqrt{\varepsilon}$; the cycles are defined by formulae (8.5)–(8.7).

To investigate the stability of the local periodic motions, one must formulate the variational equations for the family of periodic motions defined by formulae (8.2) or formulae (8.5)–(8.6), and then the characteristic equation. This cumbersome equation will not be given here. Suffice it to say that, for a symmetric solution, the equation always has one zero root and a pair of roots with opposite signs: $\rho = \pm\mu^2\omega^{-1}\rho_*$. Hence, depending on the sign of the cycle ρ_*^2, the periodic motion is either stable or hyperbolic in nature.

9. THE MODEL OF AN ELASTIC ROD SUBJECT TO A SERVO FORCE [17]

Let us consider a reversible mechanical system comprising two identical rods of mass m and length l connected to each other and to a fixed centre by means of ideal hinges and helical springs of stiffness c_1 and c_2. The second rod is subject, at its free end, to a servo force of fixed magnitude, directed along its axis. The non-deformed state of the rods corresponds to the rectangular configuration of the system. The system is in a horizontal plane.

The motion of the system is described by Lagrange's equations in which the generalized coordinates are the angles φ_1 and φ_2, namely, the deviations of the springs from the equilibrium state. Expanding the right-hand sides of the equations in powers of φ_1 and φ_2 one finds that the quadratic terms in these expansions vanish.

The system is stable in the linear approximation [17, p. 212] if

$$a^2 \geq 27c_1c_2, \quad a = 2c_1 + 16c_2 - 5Fl \tag{9.1}$$

Computing the frequencies ω_1 and ω_2 of the linear system, we find

$$\omega_{1,2}^2 = \frac{3}{7ml^2}(a \pm \sqrt{a^2 - 27c_1c_2}), \quad \omega_1 \geq \omega_2, \quad \alpha^2 = 27\frac{c_1c_2}{a^2}$$

Then, if $\alpha = (p-1)/(p+1)$, the system admits of a $1{:}p$ resonance. If $p = 1$, one has non-simple elementary divisors, but if $p = 2$, the resonance is degenerate. Hence we deduce that Theorem 5 completely solves the problem of Lyapunov families for all resonances, with the exception of the cases $p = 1$ and $p = 3$.

Let us consider 1:3 resonance in the case of identical springs ($c_1 = c_2 = c$) and compute the coefficients of system (7.1):

$$A_{11} \approx 4.88d^2, \quad A_{12} \approx 6.34d^2, \quad B_1 \approx 2.12d^2$$

$$A_{21} \approx -3.48d^2, \quad A_{22} \approx 2.51d^2, \quad B_2 \approx 1.07d^2$$

(d^2 is a certain positive number). Thus, in system (7.2) we have

$$B_1B_2 > 0, \quad B \approx 1.61d^4, \quad A_1^* \approx -7.588, \quad A_2^* \approx 9.218$$

We will compute $A = A_1^*/A_2^* \approx -0.823$. At exact resonance and for a small detuning ($\varepsilon = \mu^{\sigma+1}$, $\sigma > 1$), we obtain the case illustrated by the dashed curves in Fig. 5b. We find that $k = 1/A_2^* \approx 0.108$. Therefore, $f(u) > k^*\sqrt{u}$ for $k^* < 0$. Consequently, there is only a family $k^* > 0$. This family contains unstable periodic motions; the family is hyperbolic.

Thus, at 1:3 resonance the rectangular configuration is unstable, and if the springs undergo a small deformation, a family of oscillations is formed. The nature of the oscillations is hyperbolic; the oscillations destroy the configuration of the system.

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REFERENCES

1. TKHAI, V. N., The Lyapunov–Poincaré method in the theory of periodic motions. *Prikl. Mat. Mekh.*, 1998, **62**, 3, 355–371.
2. TKHAI, V. N., Cycle in a system close to a resonance system. *Prikl. Mat. Mekh.*, 2004, **68**, 2, 254–272.
3. MALKINI, I. G., *Some Problems of the Theory of Non-linear Oscillations*. Gostekhizdat, Moscow, 1956.
4. TKHAI, V. N., Contribution of periodic motions of a reversible system in non-structurally stable cases. Application to the *N*-planet problem. *Prikl. Mat. Mekh.*, 1998, **62**, 1, 56–72.
5. TKHAI, V. N., Lyapunov families of periodic motions is a reversible system. *Prikl. Mat. Mekh.*, 2000, **64**, 1, 46–58.
6. SEVRYUK, M. B., *Reversible Systems* (*Lecture Notes in Mathematics*, Vol. 1211). Springer, Berlin, 1986.

7. ARNOL'D, V. I., KOZLOV, V. V., and NEISHTADT, A. I., *Mathematical Aspects of Classical and Celestial Mechanics.* URSS, Moscow, 2002.
8. ROELS, J., An extension to resonant cases of Liapunov's theorem concerning the periodic solutions near a Hamiltonian equilibrium. *J. Different. Equat.*, 1971, **9**, 2, 300–324.
9. ROELS, J., Families of periodic solutions near a Hamiltonian equilibrium when the ratio of two eigenvalues is 3. *J. Different. Equat.,* 1971, **10**, 3, 431–447.
10. SWEET, D., Periodic solutions for dynamical systems possessing a first integral in the resonant case. *J. Different. Equat.*, 1973, **14**, 1, 171–183.
11. HENRARD, J., Liapunov's center theorem for resonant equilibrium. *J. Different. Equat.*, 1973, **14**, 3, 431–441.
12. SCHMIDT, D. S., Periodic solutions near a resonant equilibrium of a Hamiltonian system. *Celest. Mech.*, 1974, **9**, 1, 81–103.
13. CESARI, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*. Springer, Berlin, 1959.
14. HALE, J. K., *Oscillations in Nonlinear Systems*. McGraw-Hill, New York, 1963.
15. TKHAI, V. N., The rotational motions of mechanical systems. *Prikl. Mat. Mekh.*, 1999, **63**, 2, 179–195.
16. TKHAI, V. N., The reversibility of mechanical systems. *Prikl. Mat. Mekh.*, 1991, **55**, 4, 578–586.
17. MERKIN, D. R., *Introduction to the Theory of Stability of Motion.* Nauka, Moscow, 1976.

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